Eleven Cepheids with Variable Spectra

Star		Maximum magnitude	Range of variation	Period	Number of spectrograms	Observed spectrum variation
			mag.	*days*		
TU	Cassiopeiae....	7.3	1.1	2.139	9	F0 to F6
SU	Cassiopeiae....	5.9	0.4	1.950	19	A8 to F5
SZ	Tauri.........	7.2	0.5	3.148	11	F4 to G2
T	Monocerotis...	6.0	0.8	27.012	6	F4 to F8
RT	Aurigae.......	5.0	0.9	3.728	12	A8 to G0
W	Geminorum...	6.4	1.3	7.916	10	F3 to G0
RS	Boötis........	8.9	1.1	0.377	13	B8 to F0
X	Sagittarii......	4.4	0.6	7.012	5	F2 to G
Y	Ophiuchi......	6.2	0.8	17.121	4	F5 to G0
RR	Lyrac........	6.8	0.9	0.567	17	B9 to F2
δ	Cephei........	3.5	0.8	5.366	21	F2 to G3

smaller section of the spectrum has been shown in an earlier communication by Adams and Shapley.

Every variable for which the present test is sufficient was found to vary in spectrum. It appears safe to infer, therefore, that all Cepheids (including the cluster-type), besides being variable in light and in velocity, vary periodically in spectral class as well.

[1] Shapley and Shapley, these PROCEEDINGS, **1**, 452 (1915); Shapley, *Ibid.*, **2**, 132 (1916) Adams and Shapley, *Ibid.*, **2**, 136 (1916).

[2] These PROCEEDINGS, **2**, 143 (1916).

ON THE LINEAR DEPENDENCE OF FUNCTIONS OF SEVERAL VARIABLES, AND CERTAIN COMPLETELY INTEGRABLE SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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The study of an ordinary homogeneous linear differential equation of the nth order leads very naturally to the definition of the Wronskian of n solutions of the equation, and thence to the general theory of the linear dependence of n functions of a single variable. This is due to the characteristic property of the said differential equation, viz., that any solution of the equation is linearly dependent upon any fundamental set of solutions. I wish in this note to give some of the results which I have obtained in generalizing the theory of linear dependence to the case of n functions of several independent variables, and also to point out the application of these results to the study of an important

class of systems of partial differential equations which are a direct generalization of the single ordinary homogenous linear differential equation of the nth order. A system of the kind referred to contains a single dependent variable and any number of independent variables, and has a fundamental system of solutions, that is a definite number of linearly independent solutions in terms of which any other solutions of the system of differential equations is expressible linearly, with constant coefficients.

The following discussion is concerned throughout with functions of p independent variables. If any of the variables be complex, we shall suppose the functions to be analytic in those variables. However, we shall state all theorems for the case in which the independent variables are real, and the functions either real or complex; the modifications which must be made if some or all of the variables be complex are easily supplied, and will need no further mention. We shall impose upon the functions no restriction other than the existence of certain partial derivatives in a certain connected p-dimensional region A of the space of the independent variables.

Let $y_1, y_2, \ldots, y_n$ be functions of the p independent variables $u_1, u_2, \ldots, u_p$. We shall denote by $y_i^{(1)}$, $y_i^{(2)}$, etc., partial derivatives of y_i, of any kind or order whatever. It will be unnecessary to specify just what derivative of y_i is denoted by $y_i^{(j)}$. However, in any given discussion the same superscript (j) will denote the same derivative throughout. If a derivative $y^{(j)}$ exists for each one of the set of functions y_i $(i = 1, 2, \ldots, n)$, we shall say that *the set of functions possesses that derivative.* We may now state the fundamental theorem concerning the linear dependence of functions of several variables:

THEOREM I. *Let the set of n functions $y_1, y_2, \ldots, y_n$ of the p independent variables $u_1, u_2, \ldots, u_p$ possess enough partial derivatives, of any orders whatever, to form a matrix.*

$$M \equiv \left\| \begin{array}{llll} y_1, & y_2, & \ldots, & y_n \\ y_1^{(1)}, & y_2^{(1)}, & \ldots, & y_n^{(1)} \\ y_1^{(2)}, & y_2^{(2)}, & \ldots, & y_n^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-2)}, & y_2^{(n-2)}, & \ldots, & y_n^{(n-2)} \end{array} \right\|$$

of n–1 rows and n columns, in which at least one of the $(n$–$1)$-rowed determinants, say

$$W_n \equiv \begin{vmatrix} y_1 & , y_2 & , \ldots\ldots, & y_{n-1} \\ y_1^{(1)} & , y_2^{(1)} & , \ldots\ldots, & y_{n-1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-2)}, & y_2^{(n-2)}, & \ldots\ldots, & y_{n-1}^{(n-2)} \end{vmatrix},$$

vanishes nowhere in A. Suppose, further, that all of the first derivatives of each of the elements of the above matrix M exist, and adjoin to the matrix M such of these derivatives as do not already appear in M, to form the new matrix

$$M' \equiv \begin{Vmatrix} y_1 & , y_2 & , \ldots\ldots, & y_n \\ y_1^{(1)} & , y_2^{(1)} & , \ldots\ldots, & y_n^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-2)}, & y_2^{(n-2)}, & \ldots\ldots, & y_n^{(n-2)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(q)} & , y_2^{(q)} & , \ldots\ldots, & y_n^{(q)} \end{Vmatrix}$$

which has n columns and at least n rows, so that $q \geqq n - 1$. Then if all the n-rowed determinants of the matrix M' in which the determinant W_n is a first minor vanish identically in A, the functions $y_1, y_2, \ldots, y_n$ are linearly dependent in A, and in fact

$$y_n = c_1 y_1 + c_2 y_2 + \ldots + c_{n-1} y_{n-1},$$

the c's being constants.

The proof of this theorem is very similar to the familiar one of Frobenius for functions of a single variable [Cf. M. Bôcher, *Trans. Amer. Math. Soc.*, **2**, 139–149 (1901)]. For $p = 1$, the theorem becomes a generalization of the ordinary Wronskian theorem for functions of a single variable, and includes the latter theorem as a special case.

It should be noted that for functions of several variables it is not possible to define a single determinant which may properly be called a Wronskian; however, a Wronskian may be defined for a completely integrable system of partial differential equations, of the kind mentioned above. Before giving this definition it will be convenient to state an existence theorem for the system of partial differential equations.

Let us call a set of derivatives $y, y^{(1)}, y^{(2)}, \ldots, y^{(n-1)}$ of a function y a *normal set*, if for every element $y^{(r)}$ of the set there exists in the set at least one other element $y^{(q)}$, whose order is one less than the order of $y^{(r)}$, and from which $y^{(r)}$ may be obtained by a single differentiation. The existence theorem referred to may be stated as follows:

THEOREM II. *Suppose that in the system of partial differential equations*

$$\frac{\partial y^{(j)}}{\partial u_k} = \sum_{i=0}^{n-1} a_i^{(j,k)} y^{(i)}, \qquad (j = 0, 1, \ldots, n-1;\ k = 1, 2, \ldots, p;\ y^{(0)} \equiv y)$$

the derivatives $y, y^{(1)}, y^{(2)}, \ldots, y^{(n-1)}$ *form a normal set. Suppose further that in the closed region A the coefficients* $a_i^{(j,k)}$, *which are functions of the p independent variables,* $u_1, u_2, \ldots, u_p$, *satisfy identically the integrability conditions*

$$\frac{\partial a_\nu^{(j,k)}}{\partial u_l} + \sum_{i=0}^{n-1} a_i^{(j,k)} a_\nu^{(i,l)} = \frac{\partial a_\nu^{(j,l)}}{\partial u_k} + \sum_{i=0}^{n-1} a_i^{(j,l)} a_\nu^{(i,k)},$$
$$(\nu, j = 0, 1, \ldots, n-1;\ k, l = 1, 2, \ldots, p).$$

Let $(u_1^{(0)}, u_2^{(0)}, \ldots, u_p^{(0)})$ *be any point of A, and* $y_0, y_0^{(1)}, \ldots, y_0^{(n-1)}$ *be any set of n constants. Then there exists one and only one function y of the variables* $u_1, u_2, \ldots, u_p$ *which satisfies the system of differential equations, and whose derivatives* $y, y^{(1)}, \ldots, y^{(n-1)}$ *take on respectively the preassigned constant values* $y_0, y_0^{(1)}, \ldots, y_0^{(n-1)}$ *at the point* $(u_1^{(0)}, u_2^{(0)}, \ldots, u_p^{(0)})$.

From this theorem may be inferred at once the existence of a fundamental system of n solutions, $y_1, y_2, \ldots, y_n$, such that any other solution of the system of differential equations has the form

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Moreover, any function of this form is a solution of the completely integrable system; this system is therefore a natural generalization of the ordinary homogeneous linear differential equation of the nth order.

The derivatives $y, y^{(1)}, \ldots, y^{(n-1)}$ which appear in the right-hand members of the differential equations we shall call the *primary derivatives*. We may now define the *Wronskian* of n solutions of the completely integrable system, as the determinant formed from the primary derivatives of these solutions:

$$W \equiv \begin{vmatrix} y_1, & y_2, & \ldots, & y_n \\ y_1^{(1)}, & y_2^{(1)}, & \ldots, & y_n^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)}, & y_2^{(n-1)}, & \ldots, & y_n^{(n-1)} \end{vmatrix}.$$

The Wronskian just defined has properties similar to those met with in the theory of an ordinary differential equation. Thus, it may be shown without difficulty that

$$\frac{\partial W}{\partial u_k} = W \sum_{j=0}^{n-1} a_j^{(j,k)}, \qquad (k = 1, 2, \ldots, p)$$

so that we may determine by a quadrature a function f such that

$$\frac{\partial f}{\partial u_k} = \sum_{j=0}^{n-1} a_j^{(j,k)}, \qquad (k = 1, 2, \ldots, p)$$

Therefore, *the Wronskian W of n solutions of the completely integrable system may be determined by a quadrature from the coefficients of the system, and is given by the expression*

$$W = \text{const. } e^f.$$

This is a generalization of the theorem of Abel for an ordinary homogeneous linear differential equation of the nth order.

We shall state one more theorem, the analogue of a familiar one concerning an ordinary differential equation. The completely integrable systems to which it applies are of somewhat less generality than those for which the existence theorem has been given.

THEOREM III. *Suppose the completely integrable system considered in Theorem II has in addition the following properties:*

1^0. *The set of primary derivatives is such that, if $y^{(j)}$ be any one of the set, then all the derivatives of lower order from which $y^{(j)}$ may be obtained by differentiation also belong to the set.*

2^0. *All the first derivatives of the primary derivatives exist for each of the np coefficients $a_j^{(j,k)}$ $(j = 0, 1, \ldots, n-1;\ k = 1, 2, \ldots, p)$.*

Then the system of differential equations may be transformed in but one way into a system

$$\frac{\partial \bar{y}^{(j)}}{\partial u_k} = \sum_{i=0}^{n-1} \bar{a}_i^{(j,k)}\, \bar{y}^{(i)} \qquad (j = 0, 1, \ldots, n-1;\ k = 1, 2, \ldots, p)$$

for which all of the quantities

$$\frac{\partial \bar{f}}{\partial u_k} \equiv \sum_{j=0}^{n-1} \bar{a}_j^{(j,k)}, \qquad (k = 1, 2, \ldots, p)$$

are zero, by the transformation of the dependent variable $y = \lambda y$, where

$$\lambda = \text{const. } e^{f/n}.$$

This last theorem is of interest in the method developed in recent years by Wilczynski for dealing with questions in projective differential geometry. In fact, the coefficients of the transformed system of differential equations are what he has generally called seminvariants of the original system; the theorem affords a means for calculating these seminvariants in a purely mechanical way. In Wilczynski's method, the geometric problem becomes the study of a completely integrable system of the kind we have been considering.

The results outlined above have been developed at length in a memoir which is to appear in the *Transactions of the American Mathematical Society*.

SYSTEMATIC MOTION AMONG STARS OF THE HELIUM TYPE

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Several investigators using different methods and different material have shown beyond a doubt that the stars evidence a preference for motion in two opposite directions in the sky. This does not mean that all stars move in one of the two directions, but that there is a stronger tendency for stars to move in the favored directions than in any other.

In such investigations the helium, or B-type, stars have presented considerable difficulties, since their motions are small, and since as a class they are situated at a great distance from the sun. It has seemed desirable, therefore, to devise a method whereby the preference of motion among the helium stars might be determined with some degree of confidence.

In the first place the zone in which all the helium stars lie was mapped off into twelve arbitrary divisions. For each division means were taken of the amount of proper motion in the two co-ordinates right ascension and declination, and these mean values were then subtracted from each proper motion. Thus the center of the velocity-figure was obtained. The rectangular co-ordinates were converted into polar co-ordinates and arranged in the order of their position-angles from the north pole. Then for thirty degree groups, 0°–30°, 10°–40°, etc., means were taken of the position-angles, and sums of the amount of proper motion. With the mean position-angles as abscissae and the sums of proper motion as ordinates, the results were plotted and smooth curves drawn to represent them. Figure 1 shows how well the observations can be fitted by mooth curves. It will also be noted that there is more than one maxi-